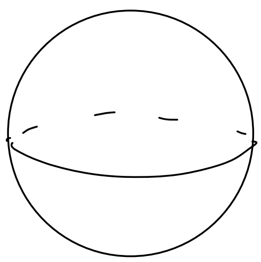
Approaches to curvature

- parallel transport around S^2
- g in normal coords
- integrability

Recall Thm Gauss curvature is intrinsic. Another way to see this:

Parallel transport

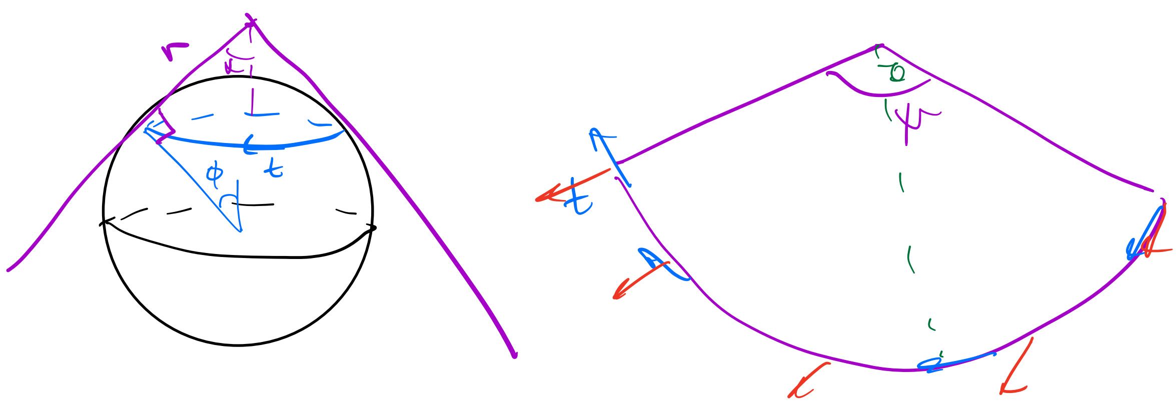
$\nabla_{\partial t} W = 0$ i.e. $\pi \bar{\nabla}_{\partial t} W = 0$ $\quad W \in C^\infty(\gamma^* TS)$



Rmk Equal for tangent surfaces.

b/c $\nabla = \pi(\bar{\nabla})$ so equal if π is the same.

Ex Latitude line of sphere



$\psi = \frac{\pi}{2} - \phi$

∠ between // vect

and so is

$2\pi - \theta$

How to find α?

$\ell(\partial B_r) = 2\pi r \sin \alpha$ $\quad\quad$ $\alpha = \frac{\pi}{2} - \phi$

$\square$ vs $\bigcirc$: connection w/ non-commuting of 2nd derivatives.

Dfn If E is a vb$/_M$ and ∇ is a connection on E, then the curvature Ω of ∇ is the End(E)-valued 2-form

$$\Omega^\nabla(X,Y)e = \nabla_X\nabla_Y e - \nabla_Y\nabla_X e - \nabla_{[X,Y]}e$$

- check 2-form (i.e. check $=0$ for trivial connection)

Calc If $A \in \Omega^1(\mathrm{End}\,E)$

$$\Omega^{\nabla+A}{}_{ij} = (\nabla_i + A_i)(\nabla_j + A_j)e - (\nabla_j + A_j)(\nabla_i + A_i)e$$

$$= \nabla_i\nabla_j - \nabla_j\nabla_i e$$
$$+ \nabla_i(A_j e) - A_j\nabla_i e$$
$$- \nabla_j(A_i e) + A_i\nabla_j e$$
$$+ (A_iA_j - A_jA_i)e$$
$$= \Omega^\nabla + (\nabla_i A_j - \nabla_j A_i + [A_i, A_j])e$$

Thm E has a local frame of flat sections wrt ∇ iff $\Omega^\nabla = 0$

$\ulcorner$ Locally, $\nabla = \partial + A$.

$$\Omega^\nabla = \partial_i A_j - \partial_j A_i + [A_i, A_j]$$

Recall (lecture 17) this is the $[\,,\,]=0$ for vector fields

$$D_i = F^a{}_b A^b_{ic}\frac{\partial}{\partial F^a{}_c} + \frac{\partial}{\partial x^i}$$

on $\mathbb{R}^n \times GL_n\mathbb{R}$ $\lrcorner$

If $E = TM$, we write $\Omega = R$ $\quad \Omega_{ij}{}^{a}{}_{b} = R_{ij}{}^{k}{}_{l}$

$$\nabla_i \nabla_j \partial_l - \nabla_j \nabla_i \partial_l = \partial_k R_{ij}{}^{k}{}_{l}$$

<u>Thm</u> $R \equiv 0 \iff M$ is locally isometric to $\mathbb{R}^n$

Pf $\Rightarrow$ $TM = \mathbb{R}^n$, $\nabla = d$ by Levi-Civita thm.

Pf $\Leftarrow$ Let E_i be a flat frame, θ^i dual frame.

$$d\langle E_i, E_j\rangle = 0 \text{ b/c metric}$$

$$d\theta^i(X,Y) = \mathrm{Alt}(\nabla\theta^i) \quad \text{by hw using torsion-free}$$
$$= 0$$

$\to$ integrate to coords y^i.

Metric tensor in these coords is δ_{ij}. $\quad \lrcorner$